The Erdős-Rényi Process Phase Transition Part II: The Fine Scaling Random Graph Processes Austin

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To me, it does not seem unlikely that on some shelf of the universe there lies a total book. I pray the unknown gods that some man - even if only one man, and though it have been thousands of years ago! - may have examined and read it. If honor and wisdom and happiness are not for me, let them be for others. May heaven exist, though my place be in hell. Let me be outraged and annihilated, but may Thy enormous Library be justified, for one instant, in one being.
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You don't have to believe in God but you should believe in The Book. – Paul Erdős

The Erdős-Rényi Processes

$G \sim G(n, p)$, $p = \frac{c}{n}$
$c < 1$ Subcritical
$c > 1$ Supercritical
$c = 1$ At Criticality, Subtle

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NO!!
The right *fine* scaling (!!)

$$p = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$$

Barely Subcritical

$\lambda \to \infty$ (but $p \sim \frac{1}{n}$)
$|C_{MAX}| = \Theta(n^{2/3}\lambda^{-2}(\ln \lambda)) = o(n^{-2/3})$
$|C_1| \sim |C_2| \sim \ldots \sim |C_k|$ for all fixed k
All components simple [1]

[1] Simple = Tree or Unicyclic

Barely Supercritical

$\lambda \to +\infty$ (but $p \sim \frac{1}{n}$)
DOMINANT COMPONENT
$|C_{MAX}| \sim 2\lambda n^{2/3} >> n^{2/3}$
C_{MAX} has high complexity [2]

[2] complexity = edges - vertices + 1

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DOMINANT COMPONENT
$|C_{MAX}| \sim 2\lambda n^{2/3} >> n^{2/3}$
C_{MAX} has high complexity [2]
Duality: Removing C_{MAX} replaces λ by $-\lambda$
$|C_2| = \Theta(n^{2/3}\lambda^{-2}(\ln \lambda)) = o(n^{-2/3})$
All other components simple.

[2] complexity = edges - vertices + 1

The Critical Window

λ fixed (positive, negative or zero)
$|C_1| = \Theta(n^{2/3})$
Size and complexity have complicated distribution.
$|C_k| = \Theta(n^{2/3})$, any fixed k
Size and complexity have complicated distribution.
Joint distribution complicated point process
No special λ – no windows inside windows

Why $n^{-4/3}$

$np = 1 + \epsilon$, $\epsilon > 0$
When finite, $|C(v)| \sim T^{PO}_{1+\epsilon}$.
Heavy tail until ϵ^{-2}, then exponential drop
All **SMALL** C have $|C| < \epsilon^{-2}$ times a bit
All "infinite" $C(v)$ join together to form dominant component.
v in dominant C with $\Pr[T^{PO}_{1+\epsilon} = \infty] \sim 2\epsilon$
DOMINANT component has size $2\epsilon n$
For dichotomy need $\epsilon^{-2} \ll 2\epsilon n$
$\epsilon \gg n^{-1/3}$

Try it!

n (perhaps 10^6) vertices. Initially: No edges
Round i: Random x_i, y_i add $\{x_i, y_i\}$
Use Union-Find for component sizes, complexity
Parametrize round e by

$$e/\binom{n}{2} = \frac{1}{n} + \lambda n^{-4/3}$$

From $\lambda = -4$ to $\lambda = +4$ see

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- $|C_2|$ rise and *fall*
- $\sum |C|^2$ becoming $|C_{MAX}|^2$

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- lots more!

Tertiary terms!

$p = n^{-1} + \lambda n^{-4/3}$, $k \sim cn^{2/3}$
$X =$ number of tree components of size k

$$E[X] = \binom{n}{k} k^{k-1} p^{k-1} (1-p)^{k(n-k)+\binom{k}{2}-(k-1)}$$

$E[X] \sim (2\pi)^{-1/2} n^{-2/3} c^{-5/2} e^{A(c)}$ with $A(c) = [(\lambda - c)^3 - \lambda^3]/6$
Density $(2\pi)^{-1/2} c^{-5/2} e^{A(c)}$
Point Process but *not* independent.
Palm process: Condition on c means $\lambda \leftarrow \lambda - c$

Asymptotic Counting

k fixed, $n \to \infty$
$C(n,k)$ = number of *connected* G, n vertices, complexity k
$\mathcal{G}$ = all such G on $\{0,\ldots,n-1\}$. $\mathcal{T}$ = trees
Cayley: $|\mathcal{T}| = C(n,0) = n^{n-2}$

$$BFS : \mathcal{G} \to \mathcal{T}$$

$C(n,k) = n^{n-2} E[BFS^{-1}(T)]$, $T \in \mathcal{T}$ uniform
$BFS^{-1}(T) = \binom{M}{k}$.
M counts $\{i,j\}$, j in queue when i popped.

BFS on Random Trees

j joins tree at time W_j, uniform
$X_t =$ number joining at time t, Poisson 1.
$Y_t =$ queue at time t
Excursion: $Y_n = 0$, $Y_t > 0$ for $t < n$.
$M = \sum_t (Y_t - 1)$, area under queue curve

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Limiting Behavior: Brownian Excursion!

$$E[\binom{M}{k}] \sim (n^{3/2})^k E[B^k]/k!$$

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$C(n,k) \sim c_k n^{n-2}[n^{3/2}]^k$
c_k Wright Constants. $c_1 = \sqrt{\pi/8}, \ldots$

More Asymptotic Counting

$p = n^{-1} + \lambda n^{-4/3}$, $k \sim cn^{2/3}$
$X^+ =$ number of components of size k
Density $(2\pi)^{-1/2} c^{-5/2} e^{A(c)} \sum_{k=0}^{\infty} c_k c^{3k/2}$

A Mysterious Process

Product Rule

Begin with no edges. Each round:

Pick x_1, x_2, y_1, y_2 at random.

IF $|C(x_1)| \cdot |C(x_2)| < |C(y_1)| \cdot |C(y_2)|$ add $\{x_1, x_2\}$ to G

ELSE add $\{y_1, y_2\}$ to G

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$f(t) = |C_{MAX}|/n$ at round $tn/2$. Critical $t_{cr} \sim 1.78$

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`Theorem: (Riordan, Warnke):` You're WRONG!

Open Questions

- What are asymptotics of $f(t_{cr} + \epsilon)$ as $\epsilon \to 0^+$

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- What are asymptotics of $f(t_{cr} + \epsilon)$ as $\epsilon \to 0^+$
- What is the scaling for the critical window
- What is $\max |C_2|$ throughout the process
- What is going on???

I like things that look difficult and intractable to solve – they challenge me because they are more interesting to figure out.
France Córdova, Director, National Science Foundation